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On the Jordan algebra and the symmetric formulation of classical mechanics

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Abstract. It is shown that, in the symmetric formulation of classical mechanics, the set of dynamical variables of the unconstrained systems constitutes a Jordan algebra under the plus Poisson bracket combination law defined by Droz-Vincent. For the constrained systems, it is shown that the set of dynamical variables constitutes a Jordan algebra under the corresponding plus Dirac bracket combination law if some conditions are satisfied. These conditions are presented.

1. Introduction

It is well known that in the standard exposition of the quantisation procedure of classical systems, the quantisation rules are

$$\{\,,\}_- \to -\frac{\mathrm{i}}{h}[\,,]_-$$

for unconstrained systems and

$$\{\,,\}^*_- \to -\frac{\mathrm{i}}{h}[\,,]^*_-$$

for constrained systems, where $\{\,,\}_-$ is the usual minus Poisson bracket, $\{\,,\}^*_-$ the minus Dirac bracket (Dirac 1950, 1951, 1958, Bergmann and Goldberg 1955) and $[\,,]_-$ and $[\,,]^*_-$ are commutators. They are valid only for integer-spin (Bose) systems.

For the unconstrained classical systems, it was shown by Droz-Vincent (1966) that, as well as the skew-symmetric algebraic structure, there exists another symmetric structure. This structure is characterised by the existence of a new bracket $\{\,,\}_+$ called the plus Poisson bracket. When we have constraints, Franke and Kálnay (1970) have shown the existence of a dual symmetric partner of the minus Dirac bracket. This new bracket, $\{\,,\}^*_+$, is called the plus Dirac bracket. In order to close an existing gap in the quantisation procedure of classical systems, Ruggeri (1974) and Kálnay and Ruggeri (1972) have suggested that, for half-integer spin (Fermi) systems, the quantisation rules are

$$\{\,,\}_+ \to \xi[\,,]_+$$

for unconstrained systems and

$$\{\,,\}^*_+ \to \xi[\,,]^*_+$$

for constrained systems, where ξ is a parameter in the theory and $[\,,\,]_+$ and $[\,,\,]^*_+$ are anticommutators.

It is also known that, for Bose-like systems, the set of classical dynamical variables of an unconstrained (constrained) system generates a Lie algebra with respect to the minus Poisson (Dirac) bracket. One concentrates on the algebraic structure (Jordan and Sudarshan 1961) of the dynamical variables in order to understand the basic mechanics of the system. Hence, in formulating the quantum theory corresponding to the mechanical system this algebraic structure of the dynamical variables is retained. In particular, it is possible to study the algebraic structure of the quantum systems with constraints from the corresponding structure of the unconstrained systems (Hermann 1969). With the new plus Poisson and Dirac brackets, it is possible, in principle, to use a similar approach for Fermi-like systems. However, the algebraic structure for the symmetric brackets has not been studied sufficiently yet.

It was pointed out by Droz-Vincent (1966) that, for systems described by plus Poisson brackets, the algebraic structure must be a Jordan algebra. However, it is not clear what the conditions are under which the classical dynamical variables constitute a Jordan algebra with respect to the plus Poisson or Dirac bracket. This is the basic purpose of the present paper. In this paper, we will not be interested in any particular classical system.

2. Notations and conventions

As in Mukunda and Sudarshan (1968), we use

$$\omega^1 = q^1, \ldots, \omega^N = q^N, \omega^{N+1} = p_1, \ldots, \omega^{2N} = p_N$$
$$\omega = (\omega^1, \omega^2, \ldots, \omega^{2N}) = (q, p).$$

With respect to the indices for coordinates in phase space, we use for q and p the indices r, s, t as in q^r; for ω we use the indices $\mu, \nu, \ldots, \pi$. For the functions ϕ which will be introduced in § 4 we shall use the indices $i, j, k, \ldots, n$. For the functions θ we reserve a, b and c. For the covariant derivative we use ∇. The local coordinates will be denoted by x^I, $I = 1, 2, \ldots, 2N$. The sum convention for any kind of indices, as well as the abbreviations $\partial_I \equiv \partial/\partial x^I$, $\partial_{IJ} \equiv \partial^2/\partial x^I \partial x^J$ and $\partial_{IJK} \equiv \partial^3/\partial x^I\, \partial x^J\, \partial x^K$, will be used systematically throughout this work.

3. The unconstrained classical systems and the Jordan algebra

Let $\mathscr{V}$ be a $2N$-dimensional manifold and $f, g, \ldots, h$ be real functions on $\mathscr{V}$. According to Droz-Vincent (1966), a plus Poisson bracket is defined by

$$\{f, M, g\}_+ \equiv \{f, g\}_+ \equiv \nabla_I (M^{JI} f) \nabla_J g = \nabla^J f \nabla_J g \tag{3.1}$$

where ∇ is the covariant derivative in a connection Γ. M is a second-rank symmetric tensor of contravariant type whose covariant derivative is zero. The functions f and g satisfy the conditions

$$\nabla_K \nabla_L \nabla_J f = 0 \qquad \nabla_K \nabla_L \nabla_J g = 0. \tag{3.2}$$

For the phase space, Droz-Vincent (1966) has shown the existence of the tensor M. Franke and Kálnay (1970) considered the connection $\Gamma = 0$ and obtained for (3.1), with the simplest choice of M, the particular form

$$\{f, g\}_+(\omega) = \epsilon_+^{\mu\nu} \partial_\mu f \partial_\nu g$$

with

$$\|\epsilon_+^{\mu\nu}\| = \begin{Vmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{Vmatrix}$$

where $\mathbb{1}$ is the $N \times N$ unit matrix.

It is then verified that the well known canonical rules for the minus Poisson brackets are also valid for the plus Poisson brackets, that is

$$\{q^r, q^s\}_+ = \{p_r, p_s\}_+ = 0 \qquad \{q^r, p_s\}_+ = \delta_s^r.$$

In the following, we consider the approach of Droz-Vincent (1966). For simplicity, we take only the case $\Gamma = 0$. However, our results can be extended to the general case $\Gamma \neq 0$.

Let us analyse, in the present section, the set F of all dynamical variables of an unconstrained classical system described by the symmetric formulation of classical mechanics. We want to obtain the conditions, if they exist, for the set F to constitute a Jordan algebra with a Jordan product binary operation (see the appendix) defined by the plus Poisson bracket combination law. Let $F \equiv (f, g, \ldots, h)$ be the set of dynamical variables. By the condition (3.2), $f, g, \ldots, h$ must be such that (we use $\Gamma = 0$)

$$\partial_{IJL} f = 0 \qquad \partial_{IJL} g = 0 \qquad \ldots \qquad \partial_{IJL} h = 0 \tag{3.3}$$

with $1 \leqslant I, J, L \leqslant 2N$. F forms a real vector space under the addition of functions and ordinary multiplication of functions by real numbers. To verify the conditions involving the $*$ product (Jordan product), we define this product by ($\Gamma = 0$)

$$f * g \equiv \{f, M, g\}_+ \equiv \{f, g\}_+ \equiv M^{KI} \partial_I f \partial_K g \tag{3.4}$$

for any f and g belonging to F.

Since the tensor M is a symmetric tensor, it is obvious that the $*$ product is commutative and in general non-associative. To analyse the identity (see the appendix)

$$(f^2 * g) * f - f^2 * (g * f) \equiv 0,$$

with

$$f^2 = f * f,$$

which must be satisfied if F is a Jordan algebra, we write for any two functions in F

$$J(f, g) = \{\{f^2, g\}_+, f\}_+ - \{f^2, \{g, f\}_+\}_+ \tag{3.5}$$

with

$$f^2 = \{f, f\}_+.$$

Then, using the relation (3.4) in (3.5) and noting that

$$\partial_I f^2 = \partial_I (f * f) = 2M^{JK} \partial_{IJ} f \partial_K f,$$

we have, with the conditions (3.3), the following results:

$$\{\{f^2, g\}_+, f\}_+ = M^{KL}M^{IJ}M^{PQ}\partial_{KP}f\partial_{QI}f\partial_{J}g\partial_{L}f + M^{KL}M^{IJ}M^{PQ}\partial_{P}f\partial_{QI}f\partial_{KJ}g\partial_{L}f$$

and

$$\{f^2, \{g, f\}_+\}_+ = M^{KL}M^{IL}M^{PQ}\partial_{P}f\partial_{QK}f\partial_{LI}g\partial_{J}f + M^{KL}M^{PQ}M^{IJ}\partial_{P}f\partial_{QK}f\partial_{I}g\partial_{LJ}f.$$

Since M is a symmetric tensor, by a change of summation indices we obtain

$$\{\{f^2, g\}_+, f\}_+ = \{f^2, \{g, f\}_+\}_+$$

or

$$J(f, g) \equiv 0.$$

It is also verified that if f and $g \in F$, then

$$\partial_{IJL}(f * g) = \partial_{IJL}\{f, g\}_+ = 0.$$

Hence F is closed under the Jordan product (3.4).

We are thus led to consider the functions $f, g, \ldots, h$, which satisfy the conditions (3.3), as forming a Jordan algebra with respect to the plus Poisson bracket.

4. Constrained classical systems and the Jordan algebra

For systems involving constraints, the Hamiltonian equations of motion can be expressed in terms of Dirac brackets in the same way in which equations of motion of unconstrained systems can be expressed in terms of Poisson brackets (Dirac 1950, 1951, 1958, Franke and Kálnay 1970). Before the introduction of the Dirac brackets, the constraints have to be separated in two classes: the first- and second-class constraints (Dirac 1950, 1951, 1958). Let us designate $\xi_a(q, p)$ as any one of the $\mathcal{M}$ constraints of a classical system. We say that ξ_a is a symmetric first-class constraint if

$$\{\xi_a, \xi_b\}_+ = 0 \qquad \text{for all } b, 1 \leqslant a, b \leqslant \mathcal{M},$$

and that ξ_a is a symmetric second-class constraint if there exists a b such that

$$\{\xi_a, \xi_b\}_+ \neq 0 \qquad 1 \leqslant a, b \leqslant \mathcal{M}.$$

Let $\Theta \equiv \{\theta_1(\omega), \theta_2(\omega), \ldots, \theta_c(\omega), \ldots, \theta_{N_\theta}(\omega)\}$ be the set of the symmetric second-class constraints of a classical system. Consider now a set Φ with $2N - N_\theta$ independent functions, $\phi_1(\omega), \ldots, \phi_m(\omega), \phi_n(\omega), \ldots, \phi_{2N-N_\theta}(\omega)$, and such that $\Psi = \Phi \cup \Theta$ is a local coordinate system for the phase-space manifold. If we denote the elements of Ψ by $\psi_1, \psi_2, \ldots, \psi_\rho, \psi_\sigma, \ldots, \psi_{2N}$, it is then shown that (Franke and Kálnay 1970)

$$\overset{+}{L}_{\rho\sigma}(\omega) = \overset{+}{\epsilon}_{\mu\nu} \frac{\partial\omega^\mu}{\partial\psi^\rho} \frac{\partial\omega^\nu}{\partial\psi^\sigma}$$

is a second-rank symmetric tensor of covariant type where

$$\|\overset{+}{\epsilon}_{\mu\nu}\| = \begin{Vmatrix} \mathbb{O} & \mathbb{1} \\ \mathbb{1} & \mathbb{O} \end{Vmatrix}$$

and $\mathbb{1}$ is the $N \times N$ unit matrix.

It follows (Franke and Kálnay 1970) that $\overset{+}{L}_{mn}$ has an inverse tensor M_+^{mn} with covariant derivatives restricted to the submanifold whose local coordinate system is Φ.

It is also possible to define the plus Dirac bracket by ($\Gamma = 0$)

$$\{f, g\}^*_+(\theta, \phi) = M^{mn}_+(\theta, \phi)\frac{\partial f}{\partial \phi^m}\frac{\partial g}{\partial \phi^n}. \tag{4.1}$$

In this approach, we use the fact that any function of the variables ω^μ can be written as a function of the variables θ^a and ϕ^m. The partial differentiation with respect to a ϕ^m is carried out keeping the θ^a's and the other ϕ^m's constant.

Notice the resemblance between (4.1) and (3.4). The difference is that in place of x^I ($I = 1, 2, \ldots, 2N$) in (3.4), we have the ϕ^m ($m = 1, 2, \ldots, 2N - N_\theta$) in (4.1), and M^{IJ} is replaced by $M^{mn}_+(\theta, \phi)$, whose derivatives are not necessarily zero.

Let us consider now a constrained classical system described by the symmetric formulation of classical mechanics. Let $\mathscr{F}$ be the corresponding set of dynamical variables ($f(\theta, \phi), g(\theta, \phi), \ldots, h(\theta, \phi)$) such that

$$\partial_{lmn} f = 0 \qquad \partial_{lmn} g = 0 \qquad \ldots \qquad \partial_{lmn} h = 0 \tag{4.2}$$

with $1 \leqslant l, m, n \leqslant 2N - N_\theta$, where

$$\partial_{lmn} = \partial^3 / \partial\phi^l \partial\phi^m \partial\phi^n.$$

We want to know if $\mathscr{F}$ is a Jordan algebra with the Jordan product defined by the plus Dirac bracket (4.1). It is obvious that $\mathscr{F}$ is a real vector space with respect to the addition of functions and ordinary multiplication of functions by real numbers. If we define in $\mathscr{F}$ the $**$ product (Jordan product) by

$$\begin{aligned}\{f ** g\}(\theta, \phi) &\equiv \{f(\theta, \phi), g(\theta, \phi)\}^*_+ \\ &\equiv M^{mn}_+(\theta, \phi)\partial_m f \partial_n g \equiv M^{mn}_+ \partial_m f\, \partial_n g\end{aligned} \tag{4.3}$$

where $\partial_m f = \partial f / \partial\phi^m$, it follows that this product is commutative (M_+ is a symmetric tensor) and, in general, non-associative. The axiom II (see the appendix) is easily verified. We also need to examine the identity

$$(f^2 ** g) ** f - f^2 ** (g ** f) \equiv 0$$

where

$$f^2 = f ** f$$

or, in terms of the plus Dirac brackets,

$$\{\{f^2, g\}^*_+, f\}^*_+ - \{f^2, \{g, f\}^*_+\}^*_+ \equiv 0$$

where $f^2 = \{f, f\}^*_+$.

To do this, we write

$$\mathscr{J}(f, g) = \{\{f^2, g\}^*_+, f\}^*_+ - \{f^2, \{g, f\}^*_+\}^*_+.$$

Then, from (4.3) and the result

$$\partial_m f^2 = \partial_m (f ** f) = \partial_m M^{jk}_+ \partial_j f \partial_k f + 2M^{jk}_+ \partial_{mj} f \partial_k f,$$

we find that $\mathscr{J}(f, g)$ is zero if

$$M^{jk}_+ M^{mn}_+ \partial_{kn} M^{li}_+ + M^{jk}_+ \partial_k M^{mn}_+ \partial_n M^{li}_+ - M^{kn}_+ \partial_k M^{jm}_+ \partial_n M^{li}_+ = 0 \tag{4.4}$$

and

$$2M_+^{ik}M_+^{mn}\partial_k M_+^{li}+2M_+^{in}M_+^{mk}\partial_k M_+^{li}-M_+^{im}M_+^{nk}\partial_k M_+^{lj}$$
$$+2M_+^{jk}M_+^{li}\partial_k M_+^{mn}-2M_+^{li}M_+^{kn}\partial_k M_+^{jm}=0. \tag{4.5}$$

To analyse if $\mathscr{F}$ is closed under the Jordan product binary operation (4.3), we consider the relation

$$L(\theta,\phi)=\partial_{ijk}(M_+^{mn}(\theta,\phi)\partial_m f\partial_n g)$$

where $f, g\in\mathscr{F}$, $1\leqslant i, j, k, m, n\leqslant 2N-N_\theta$.

It follows that $L(\theta,\phi)=0$ if

$$(\partial_{ijk}M_+^{mn}(\theta,\phi)+\partial_{ij}M_+^{mn}(\theta,\phi)\partial_k+\partial_{ik}M_+^{mn}(\theta,\phi)\partial_j+\partial_{kj}M_+^{mn}(\theta,\phi)\partial_i$$
$$+\partial_i M_+^{mn}(\theta,\phi)\partial_{kj}+\partial_j M_+^{mn}(\theta,\phi)\partial_{ki}+\partial_k M_+^{mn}(\theta,\phi)\partial_{ij})\partial_m f\partial_n g=0. \tag{4.6}$$

We are thus led to conclude that, for constrained systems described by the symmetric formulation of classical mechanics, the set of dynamical variables which satisfies (4.2) constitutes a Jordan algebra under the plus Dirac bracket combination (4.1) if the conditions (4.4), (4.5) and (4.6) are satisfied.

It is interesting to note that the relation (4.5) is satisfied if $M_+(\theta,\phi)$ is independent of ϕ^m ($m=1,2,\ldots,2N-N_\theta$). But $M_+(\theta)$ is a particular solution of the conditions (4.4), (4.5) and (4.6).

5. Conclusions

We have shown that, for unconstrained classical systems described by the symmetric formulation of classical mechanics, the set of all dynamical variables constitutes a real Jordan algebra with respect to ordinary addition and plus Poisson bracket combination. For constrained classical systems we have found that the set of dynamical variables is a Jordan algebra under ordinary addition and the plus Dirac bracket combination (4.1) if the conditions (4.4), (4.5) and (4.6) are satisfied.

We note an important difference between the Lie and Jordan algebraic structure for classical systems. The Lie algebraic structure appears in classical mechanics in a natural way, but for Jordan algebra it is different. In fact, Droz-Vincent's symmetric brackets are only defined for dynamical variables f such that ($\Gamma=0$) $\partial_{IJK}f=0$ for unconstrained systems ($1\leqslant I, J, K\leqslant 2N$) and $\partial_{lmn}f=0$ for constrained systems ($1\leqslant l, m, n\leqslant 2N-N_\theta$).

Consequently, in this symmetric formulation of classical mechanics, if f and g are dynamical variables, the quantity $fg=gf$ is not necessarily a dynamical variable. This result restricts the set of admissible dynamical variables in the present theory. However, our result is not worse than the usual theory with minus Poisson brackets. Indeed, as Streater (1966) has shown, the Dirac quantisation procedure for minus Poisson brackets is also only possible for a restricted set of dynamical variables.

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Appendix

In this appendix, for the sake of completeness, we present the definition of the Jordan algebra (Jordan *et al* 1934, Albert 1934, Jacobson 1949).

Definition. Let A be the set $\{a, b, c, \ldots\}$. If

(I) A is a real vector space;

(II) in A is defined a $*$ product (Jordan product) such that $(a+b)*c = a*c + b*c$; $\lambda(a*b) = (\lambda a)*b = a*(\lambda b)$ where λ is a real number;

(III) the $*$ product (Jordan product) is such that the following three conditions are satisfied:

(i) $a*b = b*a$ (commutative);

(ii) $(a*b)*c \neq a*(b*c)$ in general (non-associative);

(iii) $(a^2*b)*a = a^2*(b*a)$ where $a^2 = a*a$;

then A is a real Jordan algebra.

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